A Study Of Mathematical Modeling Of Cancer Cell Growth Using Generalized Logistic Model

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Abstract- *Mathematical modelling provides an effective framework for understanding the growth dynamics of cancer cells. In this study, Generalized Logistic Models (GLM) is used to analyses cancer cell growth and to unify several classical growth models. The Exponential, Logistic, VonBertalanffy and Richards's models are discussed and shown to arise from generalized growth assumptions under suitable parameter choices. The formulation of each model is presented along with the procedure for obtaining their analytical solutions. Using the derived solutions, growth values are computed at selected time intervals by solving numerical problems. To ensure a fair comparison, the same initial data is used for all models. Furthermore, Python programming is employed to perform numerical evaluations of the analytical solutions, allowing efficient computation and comparison of growth behavior across different models. The results reveal that the Exponential growth model exhibits the maximum spreading nature due to the absence of growth restrictions, making it suitable only for early-stage cancer growth analysis. In contrast, GLM and Richards models incorporate growth-limiting factors and provide more realistic and controlled descriptions of long-term cancer cell growth. This study highlights the importance of combining mathematical theory with computational tools for realistic modelling and analysis of cancer cell growth.*

I. INTRODUCTION

Mathematical modelling plays a vital role in understanding and predicting the behaviour of real-world systems. In particular, growth phenomena observed in biology, ecology, medicine, and social sciences are often nonlinear in nature and cannot be adequately explained using simple linear models. To address this limitation, several nonlinear growth models have been developed over time, among which the Generalized Logistic Model (GLM) occupies an important place.

The Generalized Logistic Model is an extension of the classical logistic growth model and provides a flexible framework for describing bounded growth processes. It is widely used in the study of population dynamics, tumor growth, epidemiology, ecological systems, and resource-limited biological growth. Unlike exponential growth models that predict unlimited growth, the GLM incorporates Environmental and biological constraints that restrict growth beyond a certain limit.

1.1 Background of Growth Models

The earliest mathematical description of growth is the exponential growth model, which assumes that the growth rate of a population is directly proportional to its current size. Although this model is simple and mathematically convenient, it fails to describe realistic systems, as it predicts unbounded growth over time. In real biological systems, resources such as nutrients, space, and energy are limited, which naturally slows down growth as the system evolves.

To overcome this drawback, the logistic growth model was introduced. The logistic model assumes that the growth rate decreases linearly as the population approaches a maximum sustainable level, known as the carrying capacity. While the logistic model provides a better approximation than exponential growth, it assumes a fixed symmetric S-shaped growth curve. However, empirical data from biological and medical studies often show asymmetric growth patterns that cannot be fully captured by the classical logistic model.

1.2 Emergence of the Generalized Logistic Model

The Generalized Logistic Model (GLM) was developed to overcome the rigidity of the classical logistic model by introducing an additional parameter known as the shape parameter. This parameter allows the growth curve to be adjusted in order to fit a wide variety of real-world data. As a result, the GLM can represent both early rapid growth and delayed growth phases more accurately than the standard logistic model.

In biological systems such as tumor growth, for example, the initial growth phase may be slow due to adaptation or limited vascularization, followed by a rapid expansion and finally a saturation phase. The classical logistic model cannot adequately capture this asymmetry, whereas the GLM provides the necessary flexibility through its generalization parameter.

1.3 Importance in Biological and Medical Applications

One of the most significant applications of the Generalized Logistic Model is in tumor growth modelling. Tumor growth is influenced by several complex factors such as nutrient supply, immune response, and cellular interactions. These factors cause the growth pattern to deviate from simple symmetric curves. The GLM allows researchers to incorporate these effects indirectly by adjusting the shape parameter, making it a powerful tool in cancer research.

In addition to tumor modelling, the GLM is extensively used in population ecology, where species growth is constrained by environmental carrying capacity. It is also applied in epidemiology to describe the spread of infectious diseases, where the number of infected individuals initially grows rapidly and later stabilizes due to immunity or intervention measures.

1.4 Mathematical Significance of GLM

From a mathematical perspective, the Generalized Logistic Model is a nonlinear ordinary differential equation that exhibits rich dynamical behaviour. The inclusion of the shape parameter introduces nonlinearity that allows for a broader class of solutions compared to the logistic model. This makes the GLM an important subject of study in applied mathematics and mathematical biology.

Furthermore, the GLM includes several well-known growth models as special cases. When the shape parameter takes a specific value, the model reduces to the classical logistic equation. For other values, it closely resembles the Richards model and other generalized growth laws. This unifying property makes the GLM a versatile and powerful modelling framework.

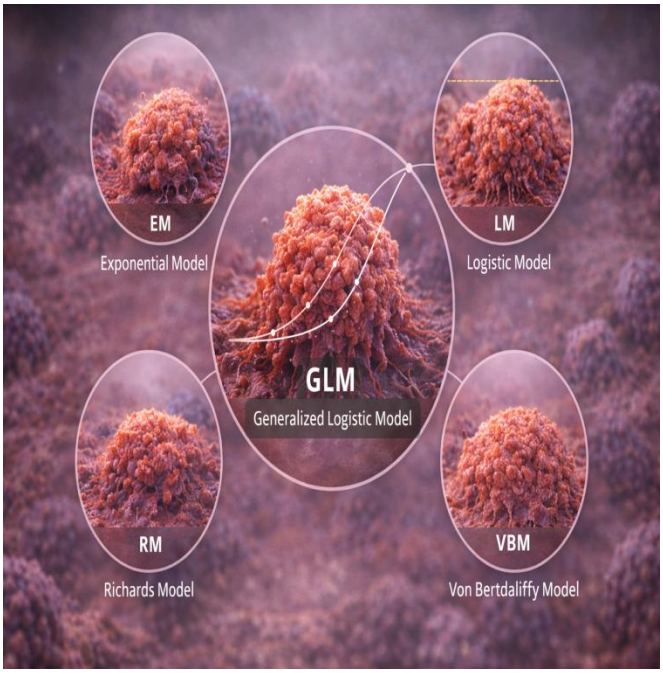
1.5 Relevance in Modern Research

In recent years, the availability of large datasets and computational tools has increased the demand for flexible mathematical models that can accurately fit experimental data. The Generalized Logistic Model has gained considerable attention in this context due to its adaptability and interpretability. Researchers can estimate model parameters using numerical and statistical methods, allowing for meaningful biological interpretations.

Moreover, the GLM serves as a foundation for developing more complex hybrid models that combine biological realism with mathematical tractability. Its ability to bridge theoretical modelling and practical data analysis makes it highly relevant in interdisciplinary research.

1.6Concluding Remarks (Intro End)

In summary, the Generalized Logistic Model represents a significant advancement in the mathematical modelling of growth processes. By extending the classical logistic model through the introduction of a shape parameter, the GLM provides a flexible and realistic description of bounded growth phenomena. Its wide range of applications in biology, medicine, ecology, and applied mathematics highlights its importance as a fundamental tool for understanding complex growth dynamics.



II. DEFINITION

2.1 Generalized Logistic Model (GLM) [ref 1,2,3,4,5,6,7]

The generalized logistic model (GLM) is unified framework that includes **Exponential**, **Logistic, Richards** and **Bertalanffy** models as special cases.

Mathematical form of the GLM is

$$\frac{dT}{dt} = PT^m \left[1 - \left(\frac{T}{K}\right)^{\delta}\right]^{\gamma}$$

Key feature

- One framework, many models
- Parameter-based flexibility

2.2 Exponential Growth model

The **Exponentialgrowthmodel** assumes that the rate of growth of population or tumor is directly proportional to its current size

Mathematical model of the **Exponential Growth model is**

$\frac{dT}{dt}$ = PT, where T (t) = tumor size at time t, p = growth rate

Key feature

- Unlimited growth
- No environmental or biological limitation

Use

- Early-stage tumor growth, bacteria growth

2.3 Logistic growth model

The **Logisticgrowthmodel** describes growth that is initially exponential but slows down as the size approaches a maximum limit called the carrying capacity.

Mathematical form of the logistic model is

$\frac{dT}{dt} = PT\left[1 - \frac{T}{K}\right]$, where T(t) tumor size of time t p=growth rate,K=carrying capacity.

Key feature

- Growth saturates at T=k
- More realistic than exponential model

Use

- Population dynamics, limited biological growth.

2.4 Von Bertalanffy growth model

The **Von Bertalanffy growth model** describes growth as the balance between anabolic (growth) and catabolic (decay) processes, making it biologically realistic.

Mathematical form of the **Von Bertalanffy growth model is**

$\frac{dT}{dt} = PT^{\frac{2}{3}}\left[1 - \left(\frac{T}{K}\right)^{\frac{1}{3}}\right]$ where p= growth rate K=carrying capacity.

Key feature

- Growth depends on surface-to-volume ratio
- Slower and realistic biological growth.

Use

- Animal growth, tissue growth, tumor modeling

2.5 Richards Growth model

The **RichardsGrowthmodel** is a generalized form of the logistic model that includes a shape parameter to control the curvature of the growth curve.

Mathematicalformofthe **Richards's growth model is**

$\frac{dT}{dt} = PT\left[1 - \left(\frac{T}{K}\right)^{\delta}\right]$ where δ=shape parameter, p=growth rate

Key feature

- Flexible growth behavior
- Includes logistic model as a special case when δ=1

Use

- Advanced biological and ecological modeling

III. DERIVATION & SOLUTION OF MATHEMATICAL MODELS OF TUMOR GROWTH [ref 5, 6, 7]

Several growth functions have been studies in epidemiology and ecology. First order ordinary differential equation (ODE) is widely used to predict tumor growth by describing changes in tumor volume over time. The most common formulation is**GeneralizedLogisticModels**

3.1 Generalized logistic model

The generalized form of the logistic equation is as follows

$\frac{dT}{dt} = PT^{m}\left[1 - \left(\frac{T}{K}\right)^{\delta}\right]^{\gamma}$ ----------> (1)

Where m,γ, and δ are nonnegative exponents and
P is the tumor growth rate in $m^3 day^{-1}$
Put m=1 γ=0 in Equation (1) we get,

$\frac{dT}{dt}$ = PT ----------> (2)

Equation (2) is called the **exponential growth model.**

Put m= γ= δ= 1 in Equation (1) we get,

$\frac{dT}{dt} = PT\left[1 - \frac{T}{K}\right]$ ----------> (3)

Equation (3) is called the **logistic growth model.**

Put m =$\frac{2}{3}$, $\delta = \frac{1}{3}$, γ = 1 in Equation (1)

$\frac{dT}{dt} = PT^{\frac{2}{3}}\left[1 - \left(\frac{T}{K}\right)^{\frac{1}{3}}\right]$ ----------> (4)

Equation (4) is called the **VonBertalanffymodel.**

Put m=1 and $\gamma = 1$ in (1)

$\frac{dT}{dt} = PT\left[1 - \left(\frac{T}{K}\right)^{\delta}\right]$----------> (5)

Equation (5) is called the **Richards growth model.**

3.2 Solution for equations (2) (3) (4) (5) models:

3.2.1 Exponential model (eqn 2)

The differential equation of the exponential model is

$$\frac{dT}{dt} = PT$$

Solution:

Separate variables

$\frac{1}{T}$ dT = P dt

Integrate on both sides

$\int \frac{1}{t} dT = \int P dt$

$\ln|T| = Pt + c$

Take "**exponential"** on both sides

$T = e^{Pt+c}$

$= e^{Pt}.e^{c}$

$T = A\, e^{Pt}$ ---------> (**A**) where $A = e^{c}$

Apply initial condition t=0, T (0) = T_0

T (0) =$A\, e^{P(0)}$

T_0=A (1)

A=T_0

Equation (**A**) becomes,

∴ **The solution of the exponential model is**

$$\mathbf{T(t) = T_0\, e^{Pt}}$$

3.2.2 Logistic model (eqn 3)

The differential equation of the logistic model is

$\frac{dT}{dt} = PT\left[1 - \frac{T}{K}\right]$

Solution:

Separate variables

$\frac{dT}{T\left(1-\frac{T}{K}\right)}$ = P dt

Partial fraction decomposition:

We decompose the left side:

$\frac{1}{T\left(1-\frac{T}{K}\right)} = \left(\frac{1}{T} + \frac{1}{K-T}\right)$

Quick check: combine the right-hand fraction:

$\frac{K}{K}\left(\frac{1}{T} + \frac{1}{K-T}\right) = \left(\frac{K-T+T}{T(K-T)}\right)$

$= \left(\frac{K-T+T}{T(K-T)}\right)$

$= \left(\frac{K}{T(K-T)}\right)$

$= \frac{1}{\frac{T(K-T)}{K}}$

$= \frac{1}{T\left(1-\frac{T}{K}\right)}$

So the identity holds. Integrate both sides

$$\int \frac{dT}{T\left(1-\frac{T}{K}\right)} = p\int dt$$

Using the decomposition $\int\left(\frac{1}{T} + \frac{1}{K-T}\right)dT = p\int dt$

$(\ln|T| - \ln|K-T|) = pt + c$

$\ln\left|\frac{T}{K-T}\right| = Pt + C$

Take "**exponential** "on both sides

$\frac{T}{K-T} = Ae^{Pt}$, A=e^{c}

Solve algebraically for T:

T= (K-T)Ae^{Pt}

⇨ $T(1+Ae^{Pt}) = kAe^{Pt}$

Thus $T(t) = \frac{KAe^{Pt}}{(1+Ae^{Pt})}$

$= \frac{\frac{KAe^{Pt}}{Ae^{Pt}}}{\frac{1+Ae^{Pt}}{Ae^{Pt}}}$

$= \frac{K}{1+\frac{1}{Ae^{Pt}}}$

$T(t) = \frac{K}{1+Be^{-Pt}}$ ------- > (B), $B = \frac{1}{A}$

If initial condition T (0) = T_0 substitute t=0

$T_0 = \frac{K}{1+Be^{-P(0)}}$

=> $1+B = \frac{K}{T_0}$

$B = \frac{K}{T_0} - 1$

$B = \frac{K-T_0}{T_0}$

Equation (**B**) becomes

∴ **The solution of logistic model is**

$$\mathbf{T(t) = \frac{k}{1+\left(\frac{K-T_0}{T_0}\right)e^{-Pt}}}$$

Where T (t) → tumor size at time t, T_0 → initial tumor size. p → Tumor growth rate. , K→ carrying capacity

3.2.3 Von Bertalanffy model (eqn 4)

The differential equation of the Bertalanffy model is

$$\frac{dT}{dt} = PT^{\frac{2}{3}}\left[1 - \left(\frac{T}{K}\right)^{\frac{1}{3}}\right]$$

Solution:

By using substitution method

$T^{\frac{1}{3}} = u$, $T^{\frac{2}{3}} = u^2$ $T = u^3$

Diff w.r.t u

$$\frac{dT}{dt} = 3u^2\frac{du}{dt}$$

From the von Bertalanffy model

$3u^2\frac{du}{dt} = P\, u^2\left[1 - \frac{u}{K^{\frac{1}{3}}}\right]$

$\frac{du}{dt} = \frac{P}{3}\left[1 - \frac{u}{K^{\frac{1}{3}}}\right]$

$\frac{du}{1-\frac{u}{K^{\frac{1}{3}}}} = \frac{P}{3}dt$

Integrating on both side

$\int \frac{1}{1-\frac{u}{K^{\frac{1}{3}}}} du = \int \frac{P}{3}\, dt$

$-K^{\frac{1}{3}}\ln\left|1 - \frac{u}{K^{\frac{1}{3}}}\right| = \frac{P}{3}t + c$

$$\ln\left|1-\frac{u}{K^{\frac{1}{3}}}\right| = \frac{P}{-3K^{\frac{1}{3}}}t + \frac{c}{-K^{\frac{1}{3}}}$$

$$\ln\left|1-\frac{u}{K^{\frac{1}{3}}}\right| = \frac{P}{-3K^{\frac{1}{3}}}t + C1 \quad (C1 = \frac{c}{-K^{\frac{1}{3}}})$$

Take "e" on both sides

$$1-\frac{u}{K^{\frac{1}{3}}} = \frac{P}{-3K^{\frac{1}{3}}}t + C1$$

$$\frac{K^{\frac{1}{3}}-u}{K^{\frac{1}{3}}} = \frac{Pt}{-3K^{\frac{1}{3}}} + C1$$

$$K^{\frac{1}{3}} - u = K^{\frac{1}{3}}\left(A\, e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)$$

$$u = K^{\frac{1}{3}} - K^{\frac{1}{3}}\left(A\, e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)$$

$$\therefore u = K^{\frac{1}{3}}\left(1 - A\, e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)$$

We know that $T = u^3$

$$T(t) = \left(K^{\frac{1}{3}}\left(1 - Ae^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)\right)^3 \rightarrow (\mathbf{C})$$

Applying the initial condition t= 0 and T (0) = T_0

$$T_0 = \left(K^{\frac{1}{3}}\left(1 - Ae^{\frac{P(0)}{-3K^{\frac{1}{3}}}}\right)\right)^3$$

$$T_0 = [K^{\frac{1}{3}}(1-A)]^3$$

$$\frac{T_0}{K} = (1-A)^3$$

$$1 - A = \left(\frac{T_0}{K}\right)^{\frac{1}{3}}$$

$$A = 1-\left(\frac{T_0}{K}\right)^{\frac{1}{3}}$$

From equation (**C**) becomes

$$T(t) = \left(K^{\frac{1}{3}}\left(1-\left(1-\left(\frac{T_0}{K}\right)^{\frac{1}{3}}\right)e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)\right)^3$$

∴The solution of Von Bertalanffy model is

$$\mathbf{T(t) = K\left(\left(1-\left(1-\left(\frac{T_0}{K}\right)^{\frac{1}{3}}\right)e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)\right)^3}$$

Where T (t) →Tumor size at time t T_0→ Initial tumor size. P→ Tumor growth rate→ Carrying capacity

3.2.4 Richards model (eqn 5)

The differential equation of the Richards model is

$$\frac{dT}{dt} = PT\left[1-\left(\frac{T}{K}\right)^{\delta}\right]$$

Solution:

$$\frac{dT}{dt} = P\left[T - T\left(\frac{T}{K}\right)^{\delta}\right]$$

$$\frac{1}{T^{1+\delta}}\frac{dT}{dt} = P\left(\frac{1}{T^{\delta}} - \frac{1}{K^{\delta}}\right)$$

Recognize derivative

$$\frac{d}{dt}\left(T^{-\delta}\right) = -\delta T^{-(\delta+1)}\frac{dT}{dt}$$

So

$$\frac{1}{-\delta}\frac{d}{dt}T^{-\delta} = \frac{1}{T^{1+\delta}}\frac{dT}{dt}$$

Substitute into equation

$$-\frac{1}{\delta}\frac{d}{dt}(T^{-\delta}) = P\,(T^{-\delta} - K^{-\delta})$$

Rearrange

$$\frac{d}{dt}\left(T^{-\delta}\right) + P\delta T^{-\delta} = \frac{P^{\delta}}{K^{\delta}}$$ (linear differential equation)

Consider $\frac{dy}{dt} + ay = b$ where $y = T^{-\delta}$ $a = P\delta$ $b = \frac{P^{\delta}}{K^{\delta}}$

Integrating factor (IF) = $e^{\int a\,dt}$

Here $a = P^{\delta}$

IF = $e^{p\delta t}$

By known equation $\frac{d}{dt}\left(T^{-\delta}\right) + P\delta T^{-\delta} = \frac{P^{\delta}}{K^{\delta}}$

Multiply whole equation by IF

i.e. $e^{p\delta t}\frac{d}{dt}\left(T^{-\delta}\right) + e^{p\delta t}\,P\delta T^{-\delta} = \frac{P^{\delta}}{K^{\delta}}e^{p\delta t}$

Recall the identity

$$e^{p\delta t}\frac{dy}{dt} + e^{p\delta t}p\delta y = \frac{P^{\delta}}{K^{\delta}}e^{p\delta t}, \quad \text{since } (T^{-\delta} = y)$$

$$\frac{d}{dt}(e^{p\delta t}y) = \frac{P^{\delta}}{K^{\delta}}e^{p\delta t}$$

$$\frac{d}{dt}(e^{p\delta t}T^{-\delta}) = \frac{P^{\delta}}{K^{\delta}}e^{p\delta t}$$

Integrating with r, to t on both sides

$$\int \frac{d}{dt}\left(e^{p\delta t}T^{-\delta}\right)dt = \int \frac{P^{\delta}}{K^{\delta}}e^{p\delta t}d$$

$$e^{p\delta t}T^{-\delta} = \frac{P^{\delta}}{K^{\delta}}\frac{1}{P^{\delta}}e^{p\delta t} + c$$

$$e^{p\delta t}T^{-\delta} = \frac{1}{K^{\delta}}e^{p\delta t} + c$$

$$T^{-\delta} = \frac{\frac{1}{K^{\delta}}e^{p\delta t} + c}{e^{p\delta t}}$$

$$= \frac{\frac{1}{K^{\delta}}e^{p\delta t}}{e^{p\delta t}} + \frac{c}{e^{p\delta t}}$$

$$T^{-\delta} = \frac{1}{K^{\delta}} + ce^{-p\delta t}$$

Take reciprocal on both sides

$$\frac{1}{T^{\delta}} = \frac{1}{\frac{1}{K^{\delta}} + Ce^{-p\delta t}}$$

$$T^{\delta} = \frac{K^{\delta}}{1+CK^{\delta}e^{-p\delta t}}$$

$$(T^{\delta})^{\frac{1}{\delta}} = \left(\frac{K^{\delta}}{1+CK^{\delta}e^{-p\delta t}}\right)^{\frac{1}{\delta}}$$

$$T = \frac{K}{(1+CK^{\delta}e^{-p\delta t})^{\frac{1}{\delta}}}$$

$\therefore T(t) = K\left[1 + CK^{\delta}e^{-p\delta t}\right]^{-\frac{1}{\delta}}$ -------> (**D**)

Applying the initial value T (0) $= T_0$, $t = 0$

$T_0 = K\left(1 + K^{\delta}e^{-p\delta(0)}c\right)^{\frac{-1}{\delta}}$

$T_0 = K\left(1 + K^{\delta}c\right)^{\frac{-1}{\delta}}$

Take both sides power -δ

$T_0^{\ -\delta} = K^{-\delta}(1 + K^{\delta}C)$

Multiply by K^{δ} on both sides

$K^{\delta}T_0^{\ -\delta} = (1 + K^{\delta}C)$

Solve c $K^{\delta}T_0^{\ -\delta} - 1 = K^{\delta}C$

$(\frac{K}{T_0})^{\delta} - 1 = K^{\delta}C$

$\frac{(\frac{K}{T_0})^{\delta} - 1}{K^{\delta}} = C$

$C = \frac{1}{T_0^{\delta}} - \frac{1}{K^{\delta}}$

From equation (**D**) becomes

$T(t) = K\left[1 + \left(\frac{1}{T_0^{\delta}} - \frac{1}{K^{\delta}}\right)K^{\delta}e^{-p\delta t}\right]^{-\frac{1}{\delta}}$

$T(t) = K\left[1 + \left(\frac{K^{\delta}}{T_0^{\delta}}\right) - 1e^{-P\delta t}\right]^{-\frac{1}{\delta}}$

∴The solution of **Richards's model is**

$T(t) = K\left[1 + \left(\left(\frac{K}{T_0}\right)^{\delta} - 1\right)e^{-P\delta t}\right]^{-\frac{1}{\delta}}$

Where T (t)→ Tumor size at time t, T_0→ Initial tumor size→ carrying capacity

P→Tumor growth rate. δ→Shape (deceleration) parameter

IV. PROBLEMS

4.1 Example Problem

A tumor grows according to the **exponential** growth model. The rate of growth of the tumor at any time t is proportional to its current size.

Initially, the tumor size is 2 cm. The proportionality constant (growth rate) is 0.08 per day.

Determine the tumor size after 5 days, 10 days, and 20 days.

Solution:

We know that the solution of the exponential model is

$T(t) = T_0\, e^{Pt}$

Where T (t)→ size of tumor at time t, P→tumor growth rate, T_0→initial l tumor size.

Here T (t) = 5, 10, 20 days

P = 0.08 per day

T_0=2

Required tumor size

(I) After 5 days

$T(5) = 2e^{0.08(5)}$

$= 2\, e^{0.4}$

T (5) ≈**2.98 cm**

(ii) After 10 days

$T(10) = 2e^{0.08(10)}$

$= 2\, e^{0.8}$

T (10) ≈4.45 cm

(iii) After 20 days

$T(20) = 2e^{0.08(20)}$

$=2e^{1.6}$

T (20) ≈ 9.91 cm

Final Answer

Time (days)	Tumor size (cm)
5	2.98
10	4.45
20	9.91

4.2 Example problem 2

A tumor grows according to the **logistic** growth model .The rate of growth of the tumor at any time t is proportional to its current size. Initially, the tumor size is 2 cm and the carrying capacity 20.The proportionality constant (growth rate) is 0.08 per day.

Determine the tumor size after 5 days, 10 days, and 20 days.

Solution:

We know that the solution of the logistic model is

$$T(t) = \frac{k}{1+\left(\frac{K-T_0}{T_0}\right)e^{-Pt}}$$

Where T (t) →tumor size at time t, T_0→ initial tumor size, P→tumor growth rate.

K→ carrying capacity

Here T (t) = 5, 10, 20 days

T_0 = 2 cm,

K =20 cm,

P = 0.08 per day.

Substitute given value

$$T(t) = \frac{20}{1+\left(\frac{20-2}{2}\right)e^{-0.08t}}$$

Required tumor size

After 5 days

$$T(5) = \frac{20}{1+\left(\frac{20-2}{2}\right)e^{-0.08(5)}}$$

$$= \frac{20}{1+(9)e^{-0.4}}$$

$\approx$ 2.844 cm

After 10 days

$$T(10) = \frac{20}{1+\left(\frac{20-2}{2}\right)e^{-0.08(10)}}$$

$$= \frac{20}{1+(9)e^{-0.8}}$$

$\approx$ 3.965 cm

After 20 days

$$T(20) = \frac{20}{1+\left(\frac{20-2}{2}\right)e^{-0.08(20)}}$$

$$= \frac{20}{1+(9)e^{-1.6}}$$

$\approx$ 7.100 cm

Final Answer (table format)

Time (days)	Tumor size(cm)
5	2.844
10	3.965
20	7.100

4.3 Example problem

A tumor grows according to the **von Bertalanffy**growth model .The rate of growth of the tumor at any time t is proportional to its current size. Initially, the tumor size is 2 cm and the carrying capacity 20.The proportionality constant (growth rate) is 0.08 per day.

Determine the tumor size after 5 days, 10 days, and 20 days.

Solution:

We know that the solution of the Von Bertalanffy growth model is

$$T(t) = K\left(\left(1-\left(1-\left(\frac{T_0}{K}\right)^{\frac{1}{3}}\right)e^{\frac{Pt}{-3K^{\frac{1}{3}}}}\right)\right)^3$$

Where T (t) →Tumor size at time t,T_0→ Initial tumor size. P→Tumor growth rate,

K→ Carrying capacity

Here T (t) = 5, 10, 20 days

T_0 = 2 cm, K =20 cm,

P = 0.08 per day.

Substitute given value

$$T(t) = 20\left(\left(1-\left(1-\left(\frac{2}{20}\right)^{\frac{1}{3}}\right)e^{\frac{0.08t}{-3(20)^{\frac{1}{3}}}}\right)\right)^3$$

$$= 20\left((1-(1-0.464)e^{-0.00983t})\right)^3$$

$$T(t) = 20(1-0.536e^{-0.00983t})^3$$

Required tumor size

After 5 days

$$T(5) = 20\left(1-0.536e^{-0.00983(5)}\right)^3$$

$$e^{-0.00983(5)} = e^{-0.04915} \approx 0.952$$

$$= 20(1-0.536 \times 0.952)^3$$

T (5) $\approx$ 2.351 cm

After 10 days

$$T(10) = 20\left(1-0.536e^{-0.00983(10)}\right)^3$$

$$e^{-0.00983(10)} = e^{-0.0983} \approx 0.906$$

$$= 20(1-0.536 \times 0.906)^3$$

T (10) $\approx$ 2.721 cm

After 20 days

$$T(20) = 20\left(1-0.536e^{-0.00983(20)}\right)^3$$

$$e^{-0.00983(20)} = e^{-0.1966} \approx 0.821$$

$$= 20(1-0.536 \times 0.821)^3$$

T (15) $\approx$3.508 cm

Final answers (table format)

Time(days)	Tumor size (cm)
5	2.351
10	2.721
20	3.508

4.4 Example problem

A tumor grows according to the **Richards** growth model .The rate of growth of the tumor at any time t is proportional to its current size. Initially, the tumor size is 2 cm and the carrying capacity 20.The proportionality constant (growth rate) is 0.08 per day. Assume the Richards shape parameter is 0.5Determine the tumor size after 5 days, 10 days, and 20 days.

Solution:

We know that the solution of the Richards model is

$$T(t) = \frac{K}{\left[1 + \left(\left(\frac{K}{T_0}\right)^{\delta} - 1\right) e^{-P\delta t}\right]^{\frac{1}{\delta}}}$$

Where T (t)→ Tumor size at time t, T_0→ Initial tumor size, K→ carrying capacity

P→ Tumor growth rate, δ→ shape (deceleration) parameter

Here T (t) = 5, 10, 20 days

T_0 = 2 cm, K =20 cm, P = 0.08 per day, δ =0.5

Substitute given value

$$T(t) = \frac{20}{\left[1 + \left(\left(\frac{20}{2}\right)^{0.5} - 1\right) e^{-0.08(0.5)(t)}\right]^{\frac{1}{0.5}}}$$

$\left(\frac{20}{2}\right)^{0.5} - 1 = \sqrt{10} - 1 \approx 3.162 - 1 = 2.162$

$e^{-0.08(0.5)(t)} = e^{-0.04(t)}$

$T(t) = 20\,[1 + 2.162\, e^{-0.04(t)}]^{-2}$

Required tumor size

After 5 days

$T(5) = 20\,[1 + 2.162\, e^{-0.04(5)}]^{-2}$

$e^{-0.04(5)} = e^{-0.2} \approx 0.819$

$= 20\,[1 + 2.162 \times 0.819]^{-2}$

$T(5) \approx 2.606$ cm

After 10 days

$T(10) = 20\,[1 + 2.162\, e^{-0.04(10)}]^{-2}$

$e^{-0.04(10)} = e^{-0.4} \approx 0.670$

$= 20\,[1 + 2.162 \times 0.670]^{-2}$

$T(10) = 3.334$ cm

After 20 days

$T(20) = 20\,[1 + 2.162\, e^{-0.04(20)}]^{-2}$

$e^{-0.04(20)} = e^{-0.8} \approx 0.449$

$= 20\,[1 + 2.162 \times 0.449]^{-2}$

$T(20) \approx 5.145$ cm

Final answers (table format)

Time(days)	Tumor size (cm)
5	2.606
10	3.334
20	5.145

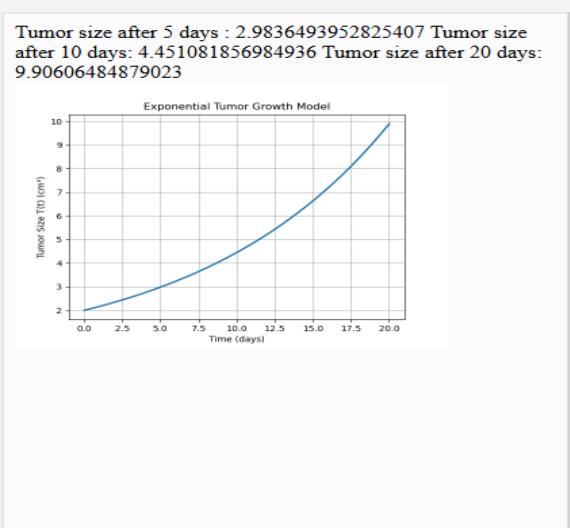
V. PYTHON PROGRAM FOR EXPONENTIAL, LOGISTIC, VON BERTALANFFY, RICHARD GROWTH MODELS (WITH OUTPUT) [ref 13]

5.1 exponential growth models

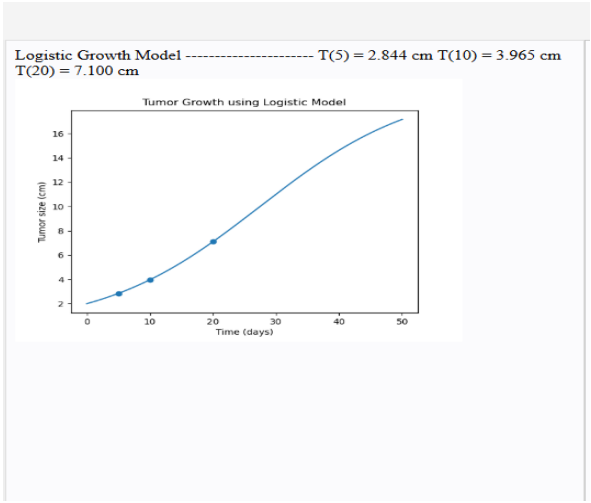
Program

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
T0 = 2       # initial tumor size (cm^3)
p = 0.08     # growth rate per day
t = np.linspace(0, 20, 100)   # time in days

# Exponential model solution
T = T0 * np.exp(p * t)

# Print some values
print("Tumor size after 5 days :", T0 * np.exp(p * 5))
print("Tumor size after 10 days:", T0 * np.exp(p *
10))
print("Tumor size after 20 days:", T0 * np.exp(p *
20))

# Plot
plt.plot(t, T, linewidth=2)
plt.xlabel("Time (days)")
plt.ylabel("Tumor Size T(t) (cm³)")
plt.title("Exponential Tumor Growth Model")
plt.grid(True)
plt.show()
```

Output



Output



5.2 logistic growth model

Program

```
import numpy as np
import matplotlib.pyplot as plt

# ------------------ Given Data ------------------
T0 = 2.0 # Initial tumor size (cm)
K = 20.0 # Carrying capacity (cm)
p = 0.08 # Growth rate (per day)

# ------------------ Logistic Model ------------------
def logistic_model(t, T0, K, p):
return K / (1 + ((K - T0) / T0) * np.exp(-p * t))

# ------------------ Time Points ------------------
times = np.array([5, 10, 20])
values = logistic_model(times, T0, K, p)

# ------------------ Output ------------------
print("Logistic Growth Model")
print("----------")
for t, v in zip(times, values):
print(f"T({t}) = {v:.3f} cm")

# ------------------ Graph ------------------
t_cont = np.linspace(0, 50, 300)
T_cont = logistic_model(t_cont, T0, K, p)

plt.figure()
plt.plot(t_cont, T_cont)
plt.scatter(times, values)
plt.xlabel("Time (days)")
plt.ylabel("Tumor size (cm)")
plt.title("Tumor Growth using Logistic Model")
plt.show()
```

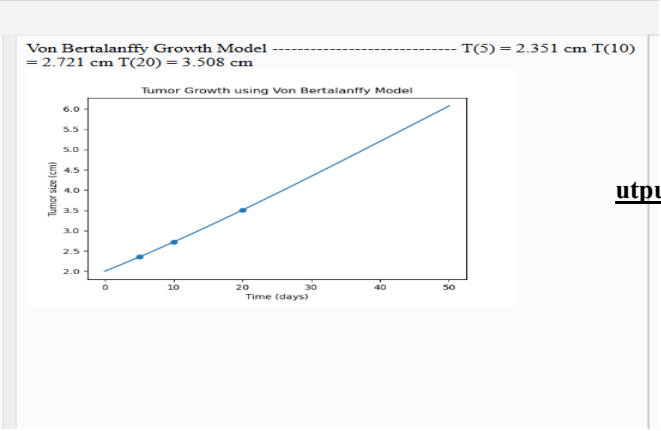
5.3 Von Bertalanffy Growth model

Program

```
import numpy as np
import matplotlib.pyplot as plt

# ------------------ Given Data ------------------
T0 = 2.0 # Initial tumor size (cm)
K = 20.0 # Carrying capacity (cm)
p = 0.08 # Growth rate (per day)

# ------------------ Von Bertalanffy Model ----------
----
def bertalanffy_model(t, T0, K, p):
return K * (1 - (1 - (T0 / K)**(1/3)) *
np.exp(-(p / (3 * K**(1/3))) * t))**3

# ------------------ Time Points ------------------
times = np.array([5, 10, 20])
values = bertalanffy_model(times, T0, K, p)

# ------------------ Output ------------------
print("Von Bertalanffy Growth Model")
print("----------")
for t, v in zip(times, values):
print(f"T({t}) = {v:.3f} cm")

# ------------------ Graph ------------------
t_cont = np.linspace(0, 50, 300)
T_cont = bertalanffy_model(t_cont, T0, K, p)

plt.figure()
plt.plot(t_cont, T_cont)
plt.scatter(times, values)
```

```
plt.xlabel("Time (days)")
plt.ylabel("Tumor size (cm)")
plt.title("Tumor Growth using Von Bertalanffy Model")
plt.show()
```

Output

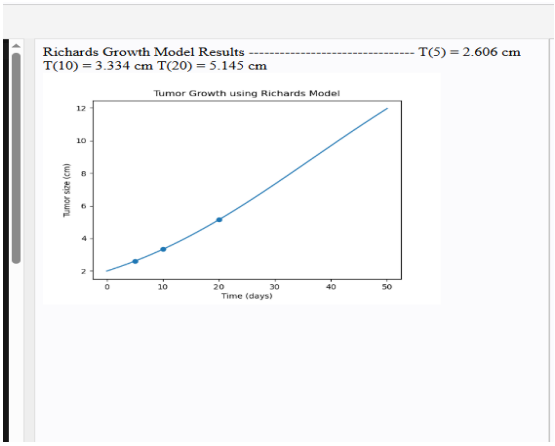


5.4 Richards growth model

Program

```
import numpy as np
import matplotlib.pyplot as plt

# ------------------ Given Parameters ------------------
T0 = 2.0 # Initial tumor size (cm)
K = 20.0 # Carrying capacity (cm)
p = 0.08 # Growth rate (per day)
m = 0.5 # Richards shape parameter

# ------------------ Richards Growth Model ------------------
def richards_model(t, T0, K, p, m):
return K * (1 + ((K / T0)**m - 1) * np.exp(-p*m*t))**(-1/m)

# ------------------ Required Time Values ------------------
times = np.array([5, 10, 20])
values = richards_model(times, T0, K, p, m)

# ------------------ Print Output ------------------
print("Richards Growth Model Results")
print("--------------------------------")
for t, v in zip(times, values):
print(f"T({t}) = {v:.3f} cm")

# ------------------ Graph ------------------
t_cont = np.linspace(0, 50, 300)
T_cont = richards_model(t_cont, T0, K, p, m)

plt.figure()
plt.plot(t_cont, T_cont)
plt.scatter(times, values)
plt.xlabel("Time (days)")
plt.ylabel("Tumor size (cm)")
plt.title("Tumor Growth using Richards Model")
plt.show()
```

Output:



VI. CONCLUSION

In this project, mathematical modeling of cancer cell growth has been studied using **GeneralizedLogisticModels** (GLM) and related growth models such as the **Exponential**, **Logistic**, **VonBertalanffy** and **Richards's** models. The study clearly explains how GLM-based models are derived from basic growth assumptions and how classical growth models emerge as special cases under appropriate parameter selections.

The project also presents the analytical solution methods for each model and demonstrates how these solutions are used to **solve numerical problems** related to cancer cell growth. Using the derived solutions, growth values were computed at different time intervals.

To support the theoretical analysis, **Python programming** was used to perform numerical computations using the same initial data for all models. This computational approach enabled effective comparison and validation of the analytical results.

The numerical and analytical results clearly indicate that the **Exponentialgrowthmodelexhibitsthemaximumspreadingnature**, as it assumes unlimited resources and predicts rapid, unbounded growth. However, this model is suitable only for the early stages of cancer cell growth. In contrast, GLM and Richards models provide controlled and realistic growth behaviour over longer periods.

Overall, this study demonstrates the formulation of GLM-based models, the process of obtaining analytical solutions, and the application of Python programming to solve growth problems. The results emphasize that while the **exponential model shows the highest spreading nature**, GLM-based models offer a more realistic framework for long-term cancer cell growth analysis.

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