Some New Results For Independent Detour Domination Number Of A Graph

A.Esakkimuthu
Dept of Mathematics
ThiyagiDharmakkanAmirtham College of Arts and Science, Kannirajapuram Post – 623 135, Ramanathapuram District, Tamilnadu, India

Abstract- *The Concept of Detour Dominating Setof a Graph was introduced in [10].A subset S of vertices in a graph G is a called a detour dominating set if S is both a detour set and a dominating set. The detour domination number* $\gamma_{dn}(G)$ *is the minimum cardinality of a detour dominating set. Any detour domination of cardinality* $\gamma_{dn}(G)$ *is called* $\gamma_{dn}-set$ *of G. In this paper we introduce the new concept of IndependentDetour Dominating Set of Graphs.A detour dominating set S of G is said to be an independentdetour dominating set of G if the sub graph induced by S is independent. The minimum cardinality among all independent detour dominating sets of G is called the independent detour domination number of G. It is denoted by* $i_{dn}(G)$. *The independent detour dominating set of cardinality* $i_{dn}(G)$*is called an* $i_{dn}-set$ *of G.In this paper, we study independent detour domination on graphs.*

Keywords- Domination,detour, detour dominating set, detour domination number, independent detour dominating set, independent detour domination number.

I. INTRODUCTION

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by *V(G)* and the edge set by *E(G)*. We define the order of *G* by $n = n(G) = |V(G)|$ and the size by $m = m(G) = |E(G)|$. For a vertex $v \in V(G)$, the *open neighborhood N(v)* is the set of all vertices adjacent to *v*, and *N[v] = N(v)*$\cup$*{v}* is the *closed neighborhood* of *v*. The *degree d(v)* of a vertex *v* is defined by *d(v)* = |*N(v)*|. The *minimum* and *maximum degrees of a graph G are* denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

For *X* $\subseteq V(G)$let *G[X]* the sub graph of *G* induced by *X*, *N(X)* = $\cup_{x \in X}$*N(x)* and *N[X]*=$\cup_{x \in X}$*N(x)*.If *G* is a connected graph, then the distance *d(x, y)* is the length of a shortest *x* − *y* path in *G*. The diameter $diam(G)$ of a connected graph is defined by *diam(G)* = $max_{x,y \in V(G)} d(x,y)$. An *x* − *y* path of length *d(x, y)* is called an *x* − *y geodesic*. A vertex *v* is said to lie on an *x* − *y* geodesic *P* if *v* is an internal vertex of *P*. The closed interval *I[x, y]* consists of *x, y* and all vertices lying on some *x* − *y* geodesic of *G*, while for $S \subseteq V(G)$, *I[S]* = $\cup_{x,y \in S} I[x,y]$. If *G* is a connected graph, then a set *S* of vertices is a geodetic *set* if *I[S]* = *V(G)*. The minimum cardinality of a geodetic set is the geodetic *number* of *G*, and is denoted by *g(G)*. The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a $g(G)-$*set*. [1, 2, 3, 4, 6].

For vertices x and y in a connected graph$G$, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in$G$. For any vertex $u$of$G$, the *detour eccentricity* of $u$is$e_D(u) = \max\{D(u,v) : v \in V\}$. A vertex v of G such that $D(u,v) = e_D(u)$ is called a *detour eccentric vertex* of u.The*detour radius*R and *detour diameter*D of $G$are defined by $R = rad_D G = \min\{e(v) : v \in V\}$ and $D = diam_D G = \max\{e(v) : v \in V\}$ respectively. An$x-y$ path of length $D(x,y)$ is called an $x-y$ *detour*. The *closed interval* $I_D[x,y]$ consists of all vertices lying on some $x-y$ detour of G, while for $S \subseteq V, I_D[S] = \cup_{x,y \in S} I_D[x,y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality$dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [3].

A vertex in a graph dominates itself and its neighbors. A set of vertices *S* in a graph *G* is a *dominating set* if *N*[*S*] = *V* (*G*). The *domination number* $\gamma(G)$ of *G* is the minimum cardinality of a dominating set of *G*. The domination number was introduced in [6].

II. INDEPENDENT DETOUR DOMINATION NUMBER

Definition 2.1:A detour dominating set S of Gis said to be an independentdetour dominating set of G if the sub graph induced by S is independent.

Example 2.2: Consider the graph G in figure 2.3

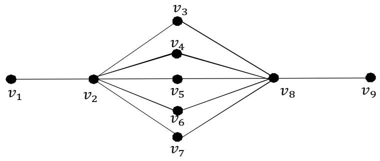


Figure 2.3

$\{v_1, v_2, v_9\}$ is a minimum detour dominating set of G and so $\gamma_{dn}(G) = 3$. Further, $\{v_1, v_3, v_4, v_5, v_6, v_7, v_9\}$ is a minimum independent detour dominating set of G and hence $i_{dn}(G) = 7$.

Observation 2.4: The following results are observed.

1. All graphs do not possess independent detour dominating set.
2. For a complete graph on p vertices, the vertex set V(G) is the detour dominating set. But it is not independent and so a complete graph has no independent detour dominating set.
3. If G contains at least two adjacent extreme vertices, then G has no independent detour dominating set.

Problem 2.5: Characterize graphs with independent detour dominating set.

Let ζ denote the collection of all graphs having at least one independent detour dominating set.

Definition 2.6: Let $G \in \zeta$, then, the minimum cardinality among all independent detour dominating sets of G is called the independent detour domination number of G. It is denoted by $i_{dn}(G)$. An independent detour dominating sets of cardinality $i_{dn}(G)$ is called an $i_{dn} - set$ of G.

Observation 2.7: Let $G \in \zeta$. The following are observed.

1. Every independent detour dominating set is an independent detour dominating set G. Therefore, $2 \le \gamma_{dn}(G) \le i_{dn}(G) \le p$.
2. Every extreme vertex of G belongs to every independent detour dominating set G.
3. Let S be the set of all extreme vertices of G. If it is an independent detour dominating set of G, then S is the unique minimum independent detour dominating set of G by observation above.
4. If G contains a clique, then any independent detour dominating set of G contains at most one vertex of the clique. Therefore, If G contains a clique, and then at most one vertex of the clique is an extreme vertex of G. For, considering the graph in figure 2.8

Let K denoted the sub graph induced by $\{v_1, v_2, v_3, v_4\}$. Then, v_1 is the unique vertex of the clique K, belonging to any detour dominating set of G.
5. The vertex of the clique belonging to any independent detour dominating set of G need not be an extreme vertex of G. For example, considering the graph in figure 2.9, let K denote the sub graph induced by $\{v_1, v_2, v_3, v_4\}$. Then, v_1 is the unique vertex of the clique K such that v_1 belongs to the minimum detour dominating set $\{v_5, v_6, v_7, v_8, v_{10}, v_1\}$ of G. But, it is not an extreme vertex of G.

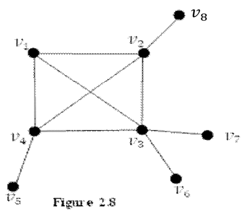


Figure 2.8

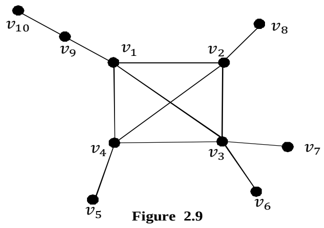


Figure 2.9

Remark 2.10: Let G be a connected graph and $G \in \zeta$. Clearly, every independent detour dominating set of G is an independent dominating set of G. Therefore, the following are true:

1. $i_{dn}(G) \ge i(G)$.
2. If S is a minimum independent detour dominating set of G, then $|V - S|$ is a dominating set of G.

Theorem 2.11: Let G be a connected graph with p vertices. Let $G \in \zeta$. Then, $i_{dn}(G) \le p - \gamma(G)$.

Proof: Let S be a minimum independent detour dominating set of G. By Remark 2.10, $\gamma(G) \le |V - S|$. Therefore, $\gamma(G) \le |V| - |S| = p - i_{dn}(G)$. Equivalently, $i_{dn}(G) \le p - \gamma(G)$.

Theorem 2.12: Let $G \in \zeta$ and let S be an independent detour dominating set of G. If $\delta(G) \ge k$, then $V - S$ is a $k - dominating\ set\ of\ G. Further,\ i_{dn}(G) \le p - \gamma_k(G)$.

Proof: Let $v \in S$. S is independent and $\delta(G) \ge k$ imply that v is adjacent to at least k vertices of $V - S$. Therefore, $V - S$ is a $k - dominating\ set\ of\ G$ and so $\gamma_k(G) \le |V - S| = |V| - |S| \le p - i_{dn}(G)$. Equivalently, $i_{dn}(G) \le p - \gamma_k(G)$.

Theorem 2.13: Let $G \in \zeta$ be a connected graphs with $p \geq 3$ vertices. Then, $i_{dn}(G) = 2$ if and only if $\gamma_{dn}(G) = 2$.

Proof: Suppose $i_{dn}(G) = 2$. Then, by Observation 2.7, $2 \leq \gamma_{dn}(G) \leq i_{dn}(G) = 2$. So, $\gamma_{dn}(G) = 2$. Conversely, Suppose $\gamma_{dn}(G) = 2$. Let $S = \{u, v\}$ be a minimum detour dominating set of G. Since $p \geq 3$, $|V - S| \neq \emptyset$. Further, every $u - v$ detour contains at least one more vertex and $d(u, v) \geq 2$. Then, $\{u, v\}$ is an independent detour dominating set of G and $i_{dn}(G) \leq 2$. By Observation 2.7, $i_{dn}(G) = 2$.

Theorem 2.15: $$i_{dn}(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n \geq 5 \\ 2 & if\ n = 3\ or\ 4 \end{cases}$$

Proof. Let $P_n = \{v_1, v_2, v_3, \ldots \ldots . v_n\}$. If n = 3 or 4, then $\{v_1, v_n\}$ is a minimum independent detour dominating set of P_n. therefore, $i_{dn}(P_3) = i_{dn}(P_4) = 2$. Let $n \geq 5$. We observe that every independent detour dominating set of P_n is a independent dominating set containing the end vertices of P_n. Let D_1 be a minimum independent dominating set containing v_1, v_n. Therefore, $|D_1| \leq |S|$. As D_1 is also a independent detour dominating set of G, $|S| \leq |D_1|$. So, we have, $i_{dn}(P_n) = |S| = |D_1|$. Let D be a minimum independent dominating set of P_n. Then,

$$|D_1| = \begin{cases} |D| = \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 1\ (mod\ 3) \\ |D| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

$$= \left\lceil \frac{n-4}{3} \right\rceil + 2. \quad \text{(By Remark 2.16)}$$

Therefore, $i_{dn}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Remark 2.16:

$$\left\lceil \frac{n-4}{3} \right\rceil + 2 = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 1\ (mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

Theorem 2.17: For $n \geq 3, i_{dn}(P_n) = \gamma_{dn}(P_n)$

Proof: Let $n \geq 3$ and let $P_n = \{v_1, v_2, \ldots \ldots . v_n\}$. If $n \equiv 0\ (mod\ 3)$ [or $n \equiv 1\ (mod\ 3)$ or $n \equiv 2\ (mod\ 3)$, then $S = \{v_1, v_4, , \ldots \ldots \ldots , v_{n-2}, v_n\}$ or $S = \{v_1, v_4, , \ldots \ldots \ldots , v_{n-3}, v_n\}$ or $S = \{v_1, v_4, , \ldots \ldots \ldots , v_{n-4}, v_{n-2}, v_n\}$ is a minimum detour dominating set of G. Also, S is independent. Therefore, $i_{dn}(P_n) \leq \gamma_{dn}(P_n)$. Hence, by Observation 2.7, $i_{dn}(P_n) = \gamma_{dn}(P_n)$.

Theorem 2.18: Let $m, n \geq 2$. Then, $i_{dn}(K_{m,n}) = \min\{m, n\}$.

Proof: Let U, W be the partition of $V(K_{m,n})$ with $|U| = m$ and $|W| = n$. Let S be an independent detour dominating set of $K_{m,n}$. Then, either $S = U\ or\ S = W$. For, if S is a proper subset of $U(\ or\ W)$, then the vertices of $U - S\ (or\ W - S)$ are not adjacent to any vertex of S. Therefore, $i_{dn}(K_{m,n}) = \min\{|U|, |W|\} = \min\{m, n\}$.

Theorem 2.19: Let G be a connected graph on p vertices. Then, $G^+ \in \zeta$ and $i_{dn}(G^+) = p$.

Proof: Let $V(G) = \{v_1, v_2, \ldots \ldots , v_p\}$ and $w_1, w_2, \ldots \ldots , w_p$ be the end vertices attached to $v_1, v_2, \ldots \ldots , v_p$ respectively in G^+. Then, $S = \{w_1, w_2, \ldots \ldots , w_p\}$ is the only minimum independent detour dominating set of G^+ and so $i_{dn}(G^+) = p$.

Proposition2.20: For a star graph G, then $i_{dn}(G) = p - 1$.

Proof: Let G = $K_{1,n}$ with $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Let S be a minimum independentdetour dominating set of $K_{1,n}$. By Observation 2.7 $\{v_1, v_2, v_3, \ldots \ldots . v_n\} \subseteq S$. Since $\{v_1, v_2, v_3, \ldots \ldots . v_n\}$ itself is a independentdetour dominating set of $K_{1,n}$, S = $\{v_1, v_2, v_3, \ldots \ldots , v_n\}$. Therefore, $i_{dn}(K_{1,n}) = n = p - 1$.

Proposition2.21: If G is a bi-star graph G, then $i_{dn}(G) = p - 2$.

Proof: Let G = $B(r, s)$ where $r, s \geq 1$. Suppose $V(B(r, s)) = \{u, v, u_i, v_i : 1 \leq i \leq r\ and\ 1 \leq j \leq s\}$ and $E(B(r, s)) = \{uv, uu_i, vv_i : 1 \leq i \leq r\ and\ 1 \leq j \leq s\}$. Let S be a minimum independentdetour dominating set of $B(r, s)$. By Observation 2.7. $\{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\} \subseteq S$. Since $\{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\}$ is itself a independentdetour dominating set of G, $S = \{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\}$. Therefore, $i_{dn}(B(r, s)) = p - 2$.

Theorem2.23: Let G be any graph with k support vertices and l end vertices. Then,
$l \leq i_{dn}(G) \leq p - k$.

Proof: LetL and K denote the set of all end and support vertices of G respectively and $|L| = l$; $|K| = k$. Clearly, $l \geq k$. By Observation 2.7, L is a subset of everyindependentdetour dominating set of G. So, $i_{dn}(G) \geq l$. Further every vertex of K lies in a double geodesic joining two vertices of L as well as independent dominated by the vertices of L. Therefore, it is clear that $V - K$ is anindependentdetour dominating set of G and so $i_{dn}(G) \leq |V - K| = |V| - |K| = p - k$. Hence the proof.

Corollary 2.24: Let T be any tree with k support vertices and l end vertices such that$l + k = p$. Then, $i_{dn}(G) = p - k$. The following example shows even if $l + k < p$, $i_{dn}(G) = p - k$.

Example2.25: consider the graph G in figure 2.23(a)

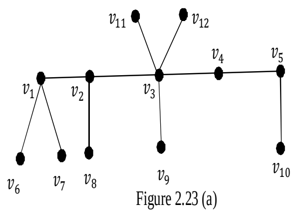
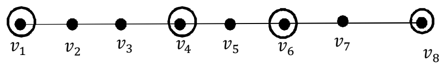


Figure 2.23 (a)

Clearly, $S = \{v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is a$i_{dn}(G) = 8 = |V - K|$.

Remark2.26: In the above theorem, both the upper and lower bounds for $i_{DG}(G')$ are sharp.

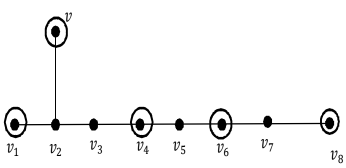
For example, consider $G = P_8, G, G'$ *and* G'' are as in figures 2.24(a), 2.24(b) and 2.24(c) respectively.



G Figure 2.24 (a)

$\{v_1, v_4, v_6, v_8\}$is a independentdetour dominating set of P_8.

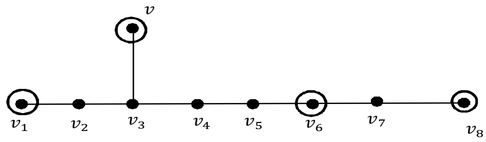
Therefore,$i_{dn}(P_8) = \left\lceil \frac{8-4}{3} \right\rceil + 2 = 2 + 2 = 4$



G' Figure 2.24 (b)

$\{v, v_1, v_4, v_6, v_8\}$is a independentdetour dominating set ofG'.

Therefore,$i_{dn}(G') = 5 = i_{dn}(P_8) + 1$.



G'' Figure 2.24 (c)

$\{v, v_1, v_5, v_8\}$is a independentdetour dominating set of G''.

Therefore,$i_{dn}(G'') = 4 = i_{dn}(P_8)$.

Theorem2.27: If a vertex is joined by an edge to any vertex ofP_n, where $n = 3k + 1$ *and* $k \geq 1$, then for the resulting graph $G' = P_n \circ K_2$, $i_{dn}(G') = i_{dn}(P_n) + 1$.

Proof:

Case 1: suppose G' is the graph obtained form P_n by adding an edge to one of the end vertices of P_n.

In this case, $G' \cong P_{n+1}$. Therefore,

$$i_{dn}(G') = i_{dn}(P_{n+1}) = \left\lceil \frac{(n+1)-4}{3} \right\rceil + 2 = \left\lceil \frac{3k+2-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-2}{3} \right\rceil + 2 = k + 2$$

and

$$i_{dn}(G) = i_{dn}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-3}{3} \right\rceil + 2 = k - 1 + 2 = k + 1.$$

Hence, $i_{dn}(G') = i_{dn}(G) + 1$.

Case 2: Suppose G' is obtained by adding an edge to one of the internal of P_n.

In this case, the number of end vertices of G' is 3. Therefore, every minimum independentdetour dominating set of G' contains these three end vertices. Clearly, any minimum independentdominating set of a path of $n - 4$ or $n - 5$ vertices along with these three end vertices forms a minimum independentdetour dominating set of G' and so $i_{dn}(G') = 3 + \left\lceil \frac{n-4}{3} \right\rceil = 3 + \left\lceil \frac{3k+1-4}{3} \right\rceil = 3 + k - 1 = k + 2 = i_{dn}(G) + 1$, as $\left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{n-5}{3} \right\rceil$ when $n = 3k + 1$.

III. CONCLUSION

Domination not only in Graph Theory but also in real life Problems plays a vital role. It helps to solve many real lifesituations.

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